The Development of the Two-Circle-Roller in a Numerical Way

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Abstract.

The Two-Circle-Roller consists of two interlocked discs. The discs are orthogonal, the distance between the centers is $r\sqrt{2}$, and r is the common radius of the discs. From this point the Two-Circle-Roller is abbreviated to TCR. The convex hull of the TCR is denoted by Ω.

The purposes of this paper are to obtain the development of the TCR, the locus of the center of gravity, the equation of Ω, the moving frame with which the rolling posture of Ω can be described, and the surface area and volume of Ω.

The development of the TCR is described with a numerical integration of the trigonometric functions, of which argument is given by the first and third kinds of elliptic integrals of Legendre-Jacobi's standard form (37), (38), (39)a, and (40)a. The numerical result is illustrated in Fig.6. (cf. Appendix 1)*

While the TCR is rolling on a plane, the height of the center of gravity O is maintained as $r/\sqrt{2}$.

The locus of the instantaneous center of the movement of O is called the Fixed Centrode, which is denoted by S_C, and is given by (68). The numerical result is shown in Fig.6.

The sphere with center O and radius $r/\sqrt{2}$ is denoted by G.

The Moving Centrode is obtained by evolving S_C on G and denoted by M_C, i.e. M_C is a closed curve on G. It is described with (66) and the numerical result is shown in Fig.7. (cf. Appendix 2)*

From vector equations (74) and (77), we obtain the equation of Ω, and the numerical result is illustrated in Fig.8. (cf. Appendix 3)*

The Moving Frame of the TCR is represented by a matrix equation (91), with which the rolling posture of the TCR on a plane can be given, and the numerical result corresponding to $t = 0.7\left\{\frac{\pi}{2} + \sin^{-1}\left(\frac{1}{1+\sqrt{2}}\right)\right\}$ is shown in Fig.11. (cf. Appendix 4)*

The surface area and volume of Ω is obtained with the first, second, and third kinds of elliptic integrals of Legendre-Jacobi's standard form (104) and (107) respectively. From numerical results (105) and (108), we can say that the surface area is 10.80% wider, and the volume is 21.65% smaller, compared with a sphere of the same radius as the TCR.

Keywords: two circle roller, development of the TCR, moving frame, moving centrode, fixed centrode, convex hull, ruled surface.

*All appendixes are source codes written with MATHEMATICA 4.1, with which we can describe the figures. The source codes are written with InputForm. Therefore if you copy the appendix from the home page URL [http://ilabo.bufsiz.jp] and paste it on MATHEMATICA with TEXT type the program will run.

1. Introduction and general description. (cf. Fig.1)

The Two-Circle-Roller consists of two interlocked discs, A and B. The discs are orthogonal, the distance between their centers is $r\sqrt{2}$, and r is the common radius of the discs. From this point the Two-Circle-Roller is abbreviated to TCR. Consider the discs to be mathematical discs, i.e. they do not have thickness. Circles C_A and C_B are the periphery of discs A and B respectively. The points O_A and O_B are the centers of C_A and C_B. The midpoint O of the line segment O_AO_B is the center of gravity (center of figure) of the TCR. The Cartesian coordinate O-xyz, which is fixed to the TCR, is defined as follows: O is the origin, the y-axis is the line OO_B, the x-axis is a line through O and perpendicular to disc B, and the z-axis is a line through O and perpendicular to disc A, with the positive direction of each axis shown by an arrow in Fig.1. The negative y-axis intersects C_A at A_O. And the positive y-axis intersects C_B at B_O. Draw a tangent line from A_O to the upper half of C_B, let B_e be the tangency point, and δ_O be $\angle B_eA_OO_B$. Draw a tangent line from B_O to the right half of C_A, and let A_e be the tangency point. Then also $\angle A_eB_OO_A = \delta_O$.

Here take point A on C_A, let t be $\angle AO_AA_O$ and s be arc-length A_OA. The tangent line of C_A at A intersects the y-axis at M. Draw a tangent line from M to the upper half of C_B and let B be the tangency point. B has symmetry point $\overline{B}$ regarding the y-axis, i.e. $\overline{B}$ is equal to B but with a negative z-coordinate. Line $M\overline{B}$ intersects the line which runs through O_A and parallel to the z-axis, because both lines belong to the yz-plane. The intersection point is denoted by K_d. The angles between the y-axis and line MB $M\overline{B}$ are denoted by δ. The plane Φ, defined by the three points M, A, $\overline{B}$ or M, A, B , is tangent to the TCR.

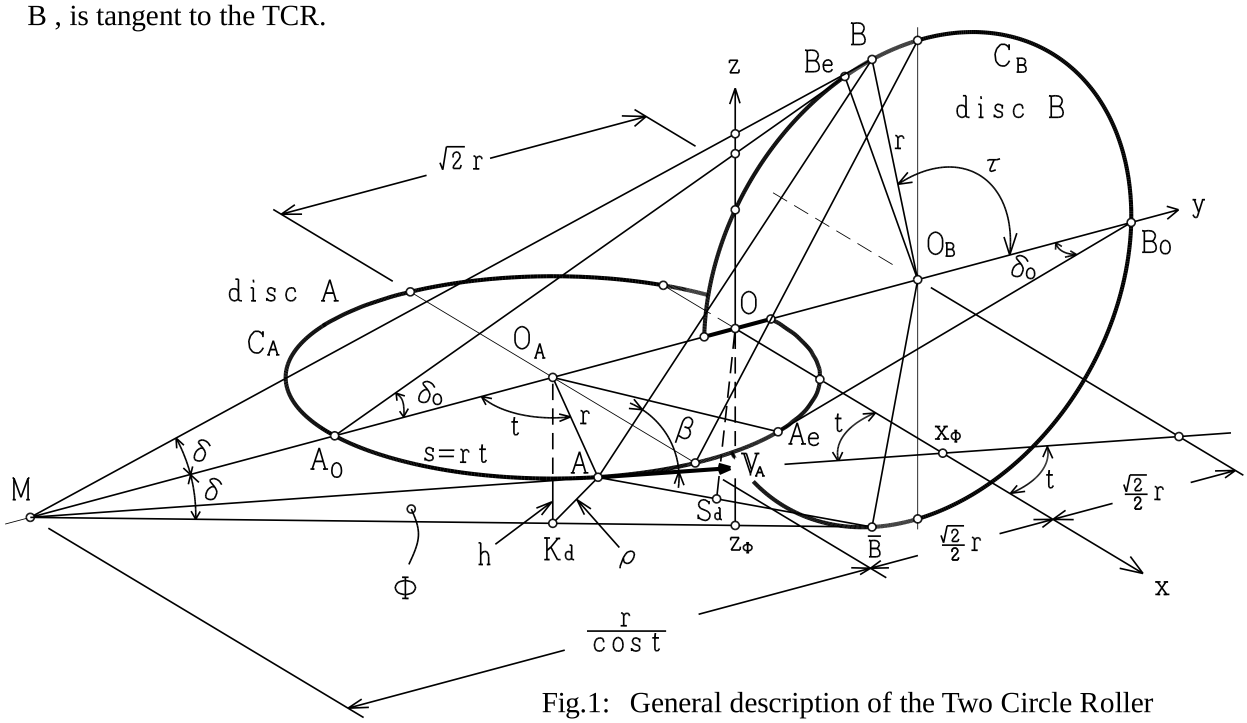

Fig.1: General description of the Two Circle Roller

From Fig.1, the equation of C_A and C_B, further coordinates of A and M are obtained as

$$C_A: \qquad x^2 + \left(y + \frac{\sqrt{2}}{2} r\right)^2 = r^2 \qquad z = 0 \qquad (1)$$

$$C_B: \qquad \left(y - \frac{\sqrt{2}}{2} r\right)^2 + z^2 = r^2 \qquad x = 0 \qquad (2)$$

$$A = \left\{ r\sin t\,, \quad -r\left(\frac{\sqrt{2}}{2} + \cos t\right), \quad 0 \right\} \tag{3}$$

$$M = \left\{ 0\,, \quad -\left(\frac{\sqrt{2}}{2} + \frac{1}{\cos t}\right), \quad 0 \right\} \quad . \tag{4}$$

Now to obtain the coordinates of B, let us compute the $\sin\delta$, and $\cos\delta$

$$\sin\delta = \frac{r}{\overline{MO_B}} = \frac{\cos t}{1+\sqrt{2}\cos t} \quad , \quad \cos\delta = \sqrt{1-\sin^2\delta} = \frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{1+\sqrt{2}\cos t} \quad . \tag{5}$$

Therefore

$$B = \left\{ 0\,, \ r\left(\frac{\sqrt{2}}{2} - \sin\delta\right), \ \pm\, r\cos\delta \right\}$$

$$= \left\{ 0\,, \ r\,\frac{\sqrt{2}}{2\left(1+\sqrt{2}\cos t\right)}\,, \ \pm r\,\frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{1+\sqrt{2}\cos t} \right\} \tag{6}$$

at (6), a negative z-coordinate means $\overline{B}$. The length AB (=A$\overline{B}$) is obtained from (3) and (6) as below:

$$\overline{AB}^2 = (0 - r\sin t)^2 + \left\{ r\,\frac{\sqrt{2}}{2\left(1+\sqrt{2}\cos t\right)} + r\left(\frac{\sqrt{2}}{2} + \cos\ t\right)\right\}^2 + \left(\pm r\,\frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{1+\sqrt{2}\cos t} - 0 \right)^2$$

$$= r^2\,\frac{2\left(\sqrt{2}+\cos t\right)^2}{1+\sqrt{2}\cos t} \quad , \tag{7}$$

then

$$\overline{AB} = \frac{r\left(2+\sqrt{2}\cos t\right)}{\sqrt{1+\sqrt{2}\cos t}} \quad . \tag{8}$$

2. Rolling posture and development of the TCR on the plane Φ

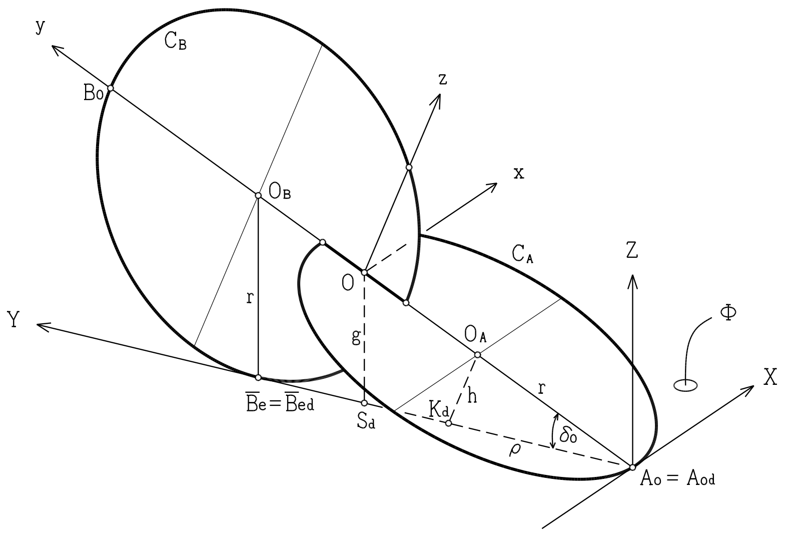

Fig.2: Initial Posture of the TCR on Φ

Set the TCR on plane Φ as shown in Fig.2, i.e. disc B is perpendicular to Φ, and $\overline{B}_e$ on C_B contacts Φ at $\overline{B}_{ed}$. Also A_O on C_A touches Φ at A_{Od} .The new Cartesian coordinate A_{Od}-XYZ is defined as that the origin is A_{Od}, the X-axis is tangent to C_A at A_O and the Y-axis is line $A_{Od}\overline{B}_{ed}$. The Z-axis is perpendicular to both X- and Y-axis and the positive direction of each axis is shown in Fig.2 with arrows. Φ is the XY-plane and S_d is the foot of the perpendicular to Φ from O. The length of line segment OS_d, i.e. the height of the center of gravity above Φ, is denoted by g.

Let the TCR wobble on Φ in a positive direction on the X-axis from the posture shown in Fig.2 without slipping. This situation is illustrated in Fig.3.

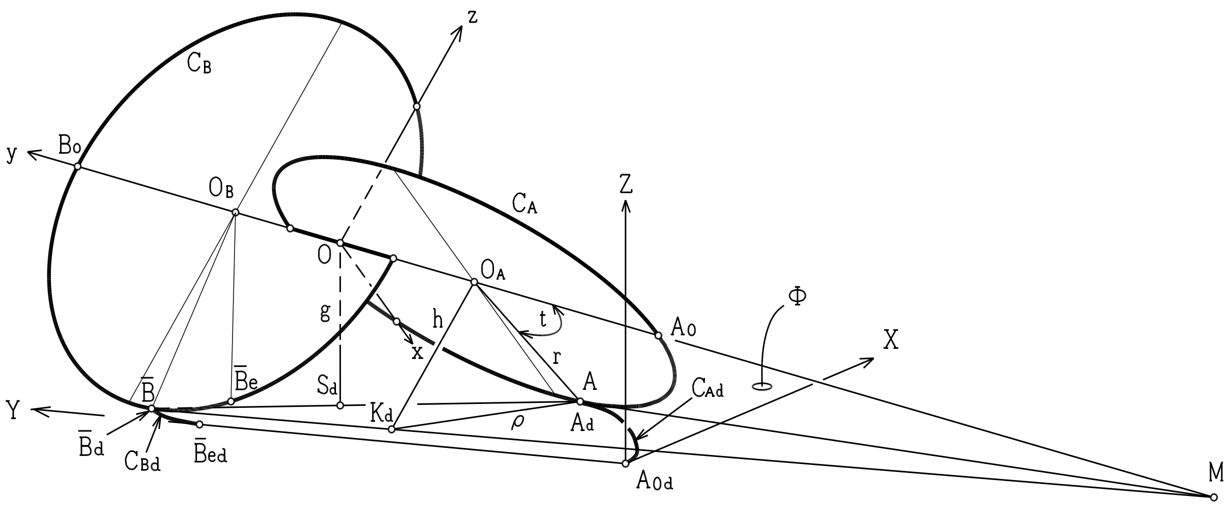

Fig.3: Wobbled posture of the TCR on Φ

The points A on C_A and $\overline{B}$ on C_B contact Φ at A_d and $\overline{B}_d$ respectively. Therefore the TCR is supported by points A_d and $\overline{B}_d$ while it is wobbling on Φ. C_{Ad} and C_{Bd} are defined as the locus of A_d and $\overline{B}_d$ respectively. C_{Ad} and C_{Bd} are two dimensional curves on Φ. The length of arc A_OA on C_A is equal to arc $A_{Od}A_d$ along C_{Ad}. Similarly the length of arc $\overline{B}_e\overline{B}$ on C_B is equal to arc $\overline{B}_{ed}\overline{B}_d$ along C_{Bd}. And all line segments $A\overline{B}$ ($A_d\overline{B}_d$), as generator lines, compose the closed convex hull Ω of the TCR. Moreover C_{Ad} and C_{Bd} are profile curves of the development obtained by evolving Ω on Φ. The tangents to C_A at A and C_{Ad} at A_d are a common tangent. This tangent intersects the y-axis at M, therefore M belongs to Φ. Let us draw a line tangent from M to C_B, and let $\overline{B}$ be the point of tangency. $\Delta M\overline{B}O_B$ lies on the yz-plane. Hence the line which passes through O_A, parallel to the z-axis, intersects line $M\overline{B}$ at K_d. The line segment AO_A (= r) is perpendicular to tangent AM. The plane which is formed with line segment AO_A and O_AK_d, is also perpendicular to AM (cf. Fig.1). Hence the line segment AK_d lies on plane AO_AK_d. Therefore line segment AK_d is also perpendicular to AM. Let h and ρ be the lengths of line segments O_AK_d and AK_d respectively, where ρ is the radius of curvature of C_{Ad} at A_d, having the following relation with r and h

$$\rho^2 = r^2 + h^2 \quad . \tag{9}$$

On the other hand, from (4), (5) and Fig.1, h can be described as

$$\begin{aligned} h &= \frac{\overline{MO_A}}{\overline{MO} + \text{coodinate of y for } \overline{B}} \ |\text{coodinate of z for } \overline{B}| \\ &= \frac{r\,\dfrac{1}{\cos t}}{r\left(\dfrac{\sqrt{2}}{2} + \dfrac{1}{\cos\ t}\right) + r\,\dfrac{\sqrt{2}}{2\left(1+\sqrt{2}\cos t\right)}} \left(r\,\frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{1+\sqrt{2}\ \cos t} \right) \\ &= \frac{r}{\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}} \quad . \end{aligned} \tag{10}$$

By substituting (10) in (9) we obtain

$$\rho^2 = \frac{r^2\left(\sqrt{2}+\cos t\right)^2}{\left(\sqrt{2}+\cos t\right)^2 - 1} \quad . \tag{11}$$

The curvature κ of C_A at A_d is the reciprocal of ρ. Then from (11) we obtain

$$\kappa = \frac{\sqrt{\left(\sqrt{2} + \cos t\right)^2 - 1}}{r\left(\sqrt{2} + \cos t\right)} \quad . \tag{12}$$

By the way, there are the following relations between length of arc s and angular parameter t:

$$s = r\,t\ ,\quad ds = r\,dt\ ,\quad t = \frac{s}{r} \quad . \tag{13}$$

Therefore κ is described as the function of s as below:

$$\kappa(s) = \frac{\sqrt{\left\{\sqrt{2} + \cos\left(\frac{s}{r}\right)\right\}^2 - 1}}{r\left\{\sqrt{2} + \cos\left(\frac{s}{r}\right)\right\}} \quad . \tag{14}$$

(14) is the two-dimensional natural-equation of C_{Ad} with arc-length s. By using (14), C_{Ad} is obtained, where $\theta(s)$ is the angle between the tangent to C_{Ad} at A_d and the X-axis (cf. Fig.4). Here A_d is defined as the point which is separated from A_{Od} by arc-length s along C_{Ad}, $\rho(s)$ is the radius of curvature of C_{Ad} at A_d , and $\theta(s)$ is the inclination angle of the tangent to C_{Ad} at A_d. Let Δs be a very short arc-length, $\rho(s+\Delta s)$ the radius of curvature of C_{Ad} at the point which is separated from A_d by Δs, and $\theta(s+\Delta s)$ the inclination angle of the tangent at $(s+\Delta s)$. By introducing the concept of mean value to $\rho(s)$, $\rho(s+\Delta s)$

$$\Delta s \cong \frac{\rho(s+\Delta s) + \rho(s)}{2}\,\Delta\theta \quad . \tag{15}$$

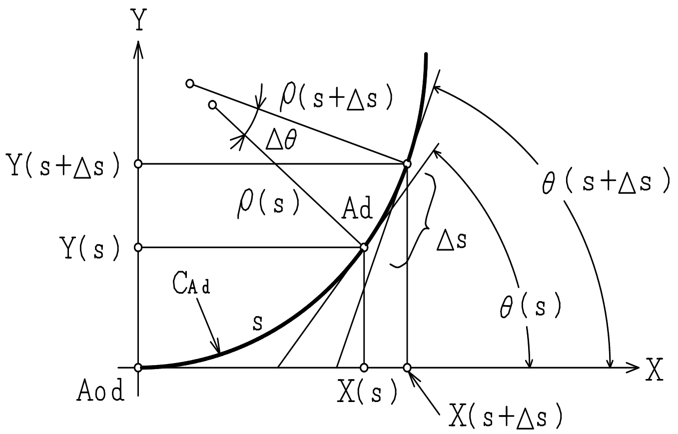

Fig.4

Therefore

$$\lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s} = \lim_{\Delta s\to 0}\frac{2}{\rho(s+\Delta s)+\rho(s)} = \frac{1}{\rho(s)} = \kappa(s) \ , \tag{16}$$

and from Fig.4 we obtain

$$\lim_{\Delta s\to 0}\frac{\theta(s+\Delta s)-\theta(s)}{\Delta s} = \lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds} \quad . \tag{17}$$

Accordingly from (16) and (17) we get

$$\frac{d\theta}{ds} = \kappa(s) \qquad \therefore\ d\theta = \kappa(s)\,ds \ . \tag{18}$$

Hence by integrating (18) under the boundary condition $\theta = 0$ at s = 0, we get

$$\theta(s) = \int_0^s \kappa(s)\,ds \ . \tag{19}$$

The coordinates X(s) and Y(s) of A_d can be calculated through the following procedure (cf.Fig.4)

$$X(s+\Delta s) \cong X(s) + \Delta s\ \cos\theta(s) \quad ,\quad Y(s+\Delta s) \cong Y(s) + \Delta s\ \sin\theta(s)\ , \tag{20}$$

hence

$$\lim_{\Delta s\to 0}\frac{X(s+\Delta s)-X(s)}{\Delta s} = \frac{dX}{ds} = \cos\theta(s) \qquad \therefore\ dX = \cos\theta(s)\,ds \tag{21}$$

$$\lim_{\Delta s\to 0}\frac{Y(s+\Delta s)-Y(s)}{\Delta s} = \frac{dY}{ds} = \sin\theta(s) \qquad \therefore\ dY = \sin\theta(s)\,ds \quad . \tag{22}$$

By integrating (21) and (22), and taking account of X (0) = Y (0) = 0, the following results can be obtained: (See [1] page at 28)

$$X(s) = \int_0^s \cos\theta(s)\, ds \tag{23}$$

$$Y(s) = \int_0^s \sin\theta(s)\, ds \quad . \tag{24}$$

By substituting (14) in (19) we get

$$\theta(s) = \int_0^s \frac{\sqrt{\left\{\sqrt{2} + \cos\left(\frac{s}{r}\right)\right\}^2 - 1}}{r\left\{\sqrt{2} + \cos\left(\frac{s}{r}\right)\right\}}\, ds \quad , \tag{25}$$

and by replacing the variable in (25) with (13), we obtain

$$\theta(t) = \int_0^t \frac{\sqrt{\left(\sqrt{2} + \cos t\right)^2 - 1}}{\sqrt{2} + \cos t}\, dt \quad . \tag{26}$$

By changing the variable in (26) with

$$\tan\frac{t}{2} = \xi \quad \therefore\ \cos t = \frac{1-\xi^2}{1+\xi^2}\ ,\ \ \sin t = \frac{2\,\xi}{1+\xi^2}\ ,\ \ \tan t = \frac{2\,\xi}{1-\xi^2}\ ,\ \ dt = \frac{2}{1+\xi^2}\, d\xi \tag{27}$$

we obtain

$$\int_0^t \frac{\sqrt{\left(\sqrt{2} + \cos t\right)^2 - 1}}{\sqrt{2} + \cos t}\, dt = \int_0^\xi \frac{\sqrt{\left(\sqrt{2} + \frac{1-\xi^2}{1+\xi^2}\right)^2 - 1}}{\left(\sqrt{2} + \frac{1-\xi^2}{1+\xi^2}\right)}\ \frac{2}{1+\xi^2}\, d\xi \quad ,$$

$$= 2\sqrt{2}\int_0^\xi \frac{\sqrt{a - b\,\xi^4}}{(1+\xi^2)(a + b\,\xi^2)}\, d\xi \tag{28}$$

where

$$a = \sqrt{2} + 1\ , \quad b = \sqrt{2} - 1 \quad \therefore\ a\,b = 1\ ,\ \frac{b}{a} = b^2 = \frac{1}{a^2} \quad . \tag{29}$$

By multiplying $\sqrt{a - b\xi^4}$ by both the denominator and numerator of the integrand of (28), we obtain the following polynomial integration:

$$= -\frac{2\sqrt{2}}{\sqrt{a}}\int_0^\xi \left\{\frac{1}{\sqrt{1 - \frac{b}{a}\,\xi^4}} - \frac{1}{(1+\xi^2)\sqrt{1 - \frac{b}{a}\,\xi^4}} - \frac{1}{\left(1 + \frac{b}{a}\,\xi^2\right)\sqrt{1 - \frac{b}{a}\,\xi^4}}\right\} d\xi\ . \tag{30}$$

Moreover, taking account of (27) and (29), by changing the variable in (30) with the following:

$$\xi = \sqrt[4]{\frac{a}{b}}\,\eta \quad \therefore\ \xi = \sqrt{a}\,\eta = \tan\frac{t}{2}\ ,\ \ \therefore\ \eta = \frac{1}{\sqrt{a}}\tan\frac{t}{2} = \sqrt{b}\ \tan\frac{t}{2} = \sqrt{\sqrt{2} - 1}\ \tan\frac{t}{2}\ ,\ \ d\xi = \sqrt{a}\, d\eta\ , \tag{31}$$

the polynomial integration (30) can be written as

$$= -\frac{2\sqrt{2}}{\sqrt{a}}\int_0^{\eta}\left\{\frac{1}{\sqrt{1-\eta^4}} - \frac{1}{(1+a\,\eta^2)\sqrt{1-\eta^4}} - \frac{1}{(1+b\,\eta^2)\sqrt{1-\eta^4}}\right\}\sqrt{a}\,d\eta \tag{32}$$

$$= -2\sqrt{2}\left\{\int_0^{\sqrt{\sqrt{2}-1}\,\tan\frac{t}{2}}\frac{1}{\sqrt{(1-\eta^2)(1+\eta^2)}}\,d\eta - \int_0^{\sqrt{\sqrt{2}-1}\,\tan\frac{t}{2}}\frac{1}{(1+a\,\eta^2)\sqrt{(1-\eta^2)(1+\eta^2)}}\,d\eta \right.$$
$$\left. - \int_0^{\sqrt{\sqrt{2}-1}\,\tan\frac{t}{2}}\frac{1}{(1+b\,\eta^2)\sqrt{(1-\eta^2)(1+\eta^2)}}\,d\eta\right\} \quad . \tag{33}$$

Now let us consider the first and third kinds of elliptic integrals of Legendre-Jacobi's standard form:

$$F(\varphi,\ k) = \int_0^{\sin\varphi}\frac{1}{\sqrt{(1-\eta^2)(1-k\,\eta^2)}}\,d\eta \tag{34}$$

$$\Pi(n\,,\varphi\,,k\,) = \int_0^{\sin\varphi}\frac{1}{(1-n\,\eta^2)\sqrt{(1-\eta^2)(1-k\,\eta^2)}}\,d\eta \quad . \tag{35}$$

By Comparing (34) with the first term in the brace of (33), and (35) with the second and third terms of (33), we obtain

$$\theta(t) = -2\sqrt{2}\left\{ F\left[\sin^{-1}\left(\sqrt{\sqrt{2}-1}\ \tan\frac{t}{2}\right), \quad -1\right]\right.$$
$$- \Pi\left[-(\sqrt{2}+1), \quad \sin^{-1}\left(\sqrt{\sqrt{2}-1}\ \tan\frac{t}{2}\right), \quad -1\right]$$
$$\left. - \Pi\left[-(\sqrt{2}-1), \quad \sin^{-1}\left(\sqrt{\sqrt{2}-1}\ \tan\frac{t}{2}\right), \quad -1\right]\right\} \quad . \tag{36}$$

That is, $\theta(t)$ can be written with elliptic integral one of the first and two of the third kind of Legendre-Jacobi's standard form. By applying the change of variable (13) to (23) and (24), the coordinates of A_d can be described with angular parameter t as below:

$$X_A(t) = r\int_0^t \cos\theta(t)\,dt \tag{37}$$

$$Y_A(t) = r\int_0^t \sin\theta(t)\,dt \quad . \tag{38}$$

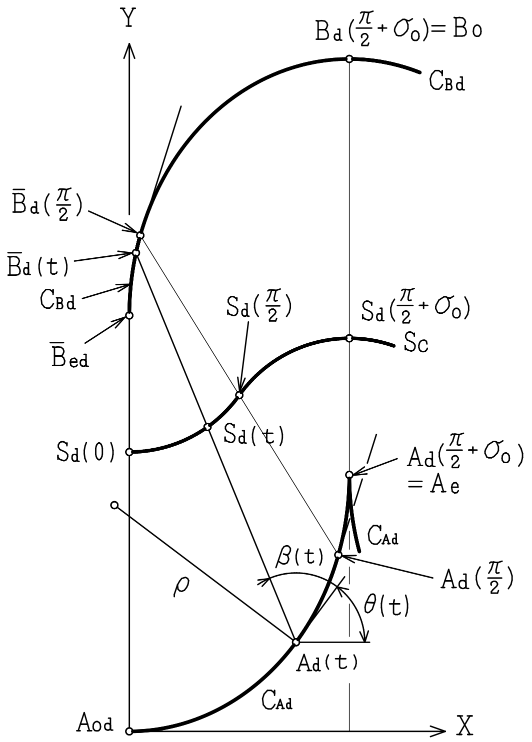

Fig.5

Further, let us work out the coordinates of $\bar{B}_d(X_B, Y_B)$ on C_B. $\beta(t)$ is an angle formed between the tangent to C_{Ad} at A_d and generator line $A_d\bar{B}_d(=A\bar{B})$. cf. Fig.1, 3, and 5. Then

$$X_B(t) = X_A(t) + \overline{A_d\bar{B}_d}\cos(\theta+\beta) \tag{39}$$

$$Y_B(t) = Y_A(t) + \overline{A_d\bar{B}_d}\sin(\theta+\beta)\ . \tag{40}$$

Here the unit tangent vector to C_{Ad} at A_d is denoted by $\mathbf{V_A}$, which is described with the O-xyz coordinate systems fixed to the TCR, letting $\mathbf{Q_A}$ be the vector of generator line $A_d\bar{B}_d$. Then, $\cos\beta$ is obtained from the inner product of $\mathbf{V_A}$ and $\mathbf{Q_A}$ as

$$\mathbf{Q_A}\cdot\mathbf{V_A} = |\mathbf{Q_A}||\mathbf{V_A}|\cos\beta\ . \tag{41}$$

Because angle t is the inclination angle of $\mathbf{V_A}$ to the x-axis as shown in Fig.1, we get

$$\mathbf{V_A} = \{\cos t\ , \quad \sin t\ , \quad 0\}\ , \tag{42}$$

and by using (3) and (6), $\mathbf{Q_A}$ can be written as

$$\mathbf{Q}_A = \left\{-r\sin t\ , \quad \sqrt{2}r\frac{1+\sqrt{2}\cos t+\cos^2 t}{1+\sqrt{2}\cos t}\ , \quad \pm r\frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2-1}}{1+\sqrt{2}\cos t}\right\}\ . \tag{43}$$

Hence the inner product of $\mathbf{Q_A}$ and $\mathbf{V_A}$ yields

$$\mathbf{Q_A}\cdot\mathbf{V_A} = \left\{-r\sin t\cos t+\sqrt{2}r\frac{1+\sqrt{2}\cos t+\cos^2 t}{1+\sqrt{2}\cos t}\sin t+0\right\} = r\sin t\frac{\sqrt{2}+\cos t}{1+\sqrt{2}\cos t}\ . \tag{44}$$

Because $\mathbf{V_A}$ is a unit vector, the absolute value of the inner product of $\mathbf{V_A}$ and $\mathbf{Q_A}$ is equal to $|\mathbf{Q_A}|$, which is obtained already in (8). For this reason from (41) we obtain

$$\cos\beta = \frac{r\sin t\dfrac{\sqrt{2}+\cos t}{1+\sqrt{2}\cos t}}{r\dfrac{\sqrt{2}\left(\sqrt{2}+\cos t\right)}{\sqrt{1+\sqrt{2}\cos t}}} = \frac{\sin t}{\sqrt{2+2\sqrt{2}\cos t}} \tag{45}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{\frac{\left(\sqrt{2}+\cos t\right)^2-1}{2+2\sqrt{2}\cos t}}\ . \tag{46}$$

By applying the addition theorem for trigonometric functions to (39) and (40), we obtain

$$X_B(t) = X_A(t) + \overline{A_d\bar{B}_d}\ \{\cos\theta\cos\beta - \sin\theta\sin\beta\} \tag{39)a}$$

$$Y_B(t) = Y_A(t) + \overline{A_d\bar{B}_d}\ \{\sin\theta\cos\beta + \cos\theta\sin\beta\}\ , \tag{40)a}$$

and by substituting (37), (38), (36), (45) and (46) in (39)a and (40)a, the X, Y coordinates of $\bar{B}_d$ on C_{Bd} can be obtained. However $\theta(t)$ is represented by the elliptic integrals (36), so it is hard to integrate

(37), (38), (39)a, and (40)a analytically. Hence they are integrated numerically with MATHEMATICA (cf. [2]). For numerical calculation, the range of t is set at $0 \le t \le \pi/2$ and divided evenly in j. In case of $t = \pi/2$, the y- coordinate of M in Fig.1 tends to infinity. However it can be forecast from Fig.3 that both disc A, and disc B make an angle $\pi/4$ with the XY-plane (= Φ) at this moment, and after this discs A and B change posture toward each other according to the wobbling of the TCR. Therefore C_{Ad} and C_{Bd} can be obtained by a point-reflection over $S_d(\pi/2)$ which is the midpoint of line segment $\overline{A_d(\pi/2)\overline{B}_d(\pi/2)}$. However the slope of tangents C_{Ad} at $A_d(\pi/2)$; m_A, and C_{Bd} at $\overline{B}_d(\pi/2)$; m_B must be equal for a smooth connection of the curves with point-reflection. This can be confirmed with the following procedure. From (37), (38) and (36) we obtain

$$m_A = \frac{dY_A(t)}{dX_A(t)} = \left\{\frac{\frac{dY_A(t)}{dt}}{\frac{dX_A(t)}{dt}}\right\}_{t=\frac{\pi}{2}} = \tan\left\{\theta\left(\frac{\pi}{2}\right)\right\}$$

$$= \tan\left\{-2\sqrt{2}\left\{F\left[\sin^{-1}\left(\sqrt{\sqrt{2}-1}\right), -1\right] - \Pi\left[-(\sqrt{2}+1), \sin^{-1}\left(\sqrt{\sqrt{2}-1}\right), -1\right]\right.\right.$$

$$\left.\left. - \Pi\left[-(\sqrt{2}-1), \sin^{-1}\left(\sqrt{\sqrt{2}-1}\right), -1\right]\right\}\right\} = 4.484256669509817\cdots \quad . \quad (47)$$

And from (39) and (40) we get

$$m_B = \frac{dY_B(t)}{dX_B(t)} = \left\{\frac{\frac{dY_B(t)}{dt}}{\frac{dX_B(t)}{dt}}\right\}_{t=\frac{\pi}{2}} = \left(\frac{\frac{dY_A(t)}{dt} + \frac{d}{dt}\left\{\overline{A_d\overline{B}_d}\sin(\theta+\beta)\right\}}{\frac{dX_A(t)}{dt} + \frac{d}{dt}\left\{\overline{A_d\overline{B}_d}\cos(\theta+\beta)\right\}}\right)_{t=\frac{\pi}{2}}$$

$$= \left(\frac{\frac{dY_A(t)}{dt} + \frac{d\left(\overline{A_d\overline{B}_d}\right)}{dt}\sin\{\theta(t)+\beta(t)\} + \left(\overline{A_d\overline{B}_d}\right)\cos\{\theta(t)+\beta(t)\}\left\{\frac{d\theta(t)}{dt} + \frac{d\beta(t)}{dt}\right\}}{\frac{dX_A(t)}{dt} + \frac{d\left(\overline{A_d\overline{B}_d}\right)}{dt}\cos\{\theta(t)+\beta(t)\} - \left(\overline{A_d\overline{B}_d}\right)\sin\{\theta(t)+\beta(t)\}\left\{\frac{d\theta(t)}{dt} + \frac{d\beta(t)}{dt}\right\}}\right)_{t=\frac{\pi}{2}} \quad . \quad (48)$$

Where from (37), (38)

$$\frac{dY_A(t)}{dt} = r\sin\theta(t)\ , \qquad \frac{dX_A(t)}{dt} = r\cos\theta(t)\ , \quad (49)$$

because of (8)

$$\frac{d\left(\overline{A_d\overline{B}_d}\right)}{dt} = \frac{d}{dt}\left\{r\frac{\sqrt{2}(\sqrt{2}+\cos t)}{\sqrt{1+\sqrt{2}\cos t}}\right\} = -\frac{r\sin t\cos t}{\left(1+\sqrt{2}\cos t\right)^{\frac{3}{2}}}\ , \quad (50)$$

due to (26)

$$\frac{d\theta(t)}{dt} = \frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{\sqrt{2}+\cos t}\ , \quad (51)$$

according to (45), (46)

$$\frac{d\{\cos\beta(t)\}}{dt} = \frac{d\{\cos\beta(t)\}}{d\beta(t)}\frac{d\beta(t)}{dt} = \frac{d}{dt}\left\{\frac{\sin t}{\sqrt{2+2\sqrt{2}\cos t}}\right\}$$

$$\therefore \quad \frac{d\beta(t)}{dt} = \frac{1}{-\sin\beta(t)}\frac{d}{dt}\left\{\frac{\sin t}{\sqrt{2+2\sqrt{2}\cos t}}\right\} = -\frac{2\cos t + \sqrt{2}(1+\cos^2 t)}{2\left(1+\sqrt{2}\cos t\right)\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}} \quad . \tag{52}$$

First substitute (49), (50), (51), (52) and (8) in (48), then expand (48) with the addition theorem for trigonometric functions, and by using (45), (46) and (36). A value m_B that corresponds to $t = \pi/2$ is obtained as

$$m_B = 4.484256669509838\cdots \quad . \tag{53}$$

By comparing (53) with (47), we can notice m_A is equal to m_B until 13 decimal places. Hence we can regard them as numerically equal. Then the curves C_{Ad} and C_{Bd} can be connected smoothly with point-reflection. Further, in Fig.5, the curves can be reflected to the right side of line segment $B_O A_e$ with line symmetry. And similarly, the curves on the positive side of the X-axis are reflected to the negative side with line symmetry over the Y-axis. Therefore the whole development curves C_{Ad} and C_{Bd} can be obtained with point and line reflections. Before showing the development curves, I think we had better discuss the movement of the center of gravity.

3. Movement of Center of Gravity O

Let the coordinates of A and $\overline{B}$ be (x_A, y_A, z_A) and $(x_{\overline{B}}, y_{\overline{B}}, z_{\overline{B}})$ with O-xyz coordinate systems respectively, then from (3) and (6)

$$x_A = r\sin t\ , \qquad y_A = -r\,\frac{\sqrt{2}+2\cos t}{2}\ , \qquad z_A = 0 \tag{54}$$

$$x_{\overline{B}} = 0\ , \quad y_{\overline{B}} = r\,\frac{\sqrt{2}}{2\left(1+\sqrt{2}\cos t\right)}\ , \quad z_{\overline{B}} = -r\,\frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{1+\sqrt{2}\cos t} \quad . \tag{55}$$

From (8) we can get the length of line segment $A\overline{B}$, then the direction cosines of $A\overline{B}$ are obtained as

$$\left.\begin{aligned}
\lambda &= \frac{(x_{\overline{B}} - x_A)}{\overline{A\overline{B}}} = -\frac{\sin t\sqrt{1+\sqrt{2}\cos t}}{2+\sqrt{2}\cos t}\\
\mu &= \frac{(y_{\overline{B}} - y_A)}{\overline{A\overline{B}}} = \frac{1+\sqrt{2}\cos t + \cos^2 t}{\left(\sqrt{2}+\cos t\right)\sqrt{1+\sqrt{2}\cos t}}\\
\nu &= \frac{(z_{\overline{B}} - z_A)}{\overline{A\overline{B}}} = -\frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{\left(2+\sqrt{2}\cos t\right)\sqrt{1+\sqrt{2}\cos t}}
\end{aligned}\right\} \quad . \tag{56}$$

Therefore the equation of line $A\overline{B}$ is given as

$$\frac{x - x_A}{\lambda} = \frac{y - y_A}{\mu} = \frac{z}{\nu} \quad . \tag{57}$$

Accordingly, the length of the perpendicular line from O to line $A\overline{B}$, i.e. g can be described as below:

$$g^2 = {x_A}^2 + {y_A}^2 - (\lambda\, x_A + \mu\, y_A)^2 = r^2\left(\frac{3}{2} + \sqrt{2}\cos t\right) - r^2\left(1+\sqrt{2}\cos\ t\right) = \frac{r^2}{2} \quad . \tag{58}$$

$$\therefore \ g = \frac{r}{\sqrt{2}} \tag{59}$$

Here let x_Φ, y_Φ and z_Φ be intercepts on the O-xyz coordinate axis with Φ, then the length of the perpendicular line from O to Φ is denoted by g_1 and it can be written as below:

$$g_1 = \frac{x_\Phi\, y_\Phi\, z_\Phi}{\sqrt{{x_\Phi}^2\, {y_\Phi}^2 + {y_\Phi}^2\, {z_\Phi}^2 + {z_\Phi}^2 {x_\Phi}^2}} \quad . \tag{60}$$

From Fig.1, the positive x-intercept is given by

$$x_\Phi = \frac{\overline{MO}}{\tan t} = r\,\frac{2+\sqrt{2}\cos t}{2\sin t} \quad . \tag{61}$$

In addition, by regarding (4) and (5), the negative z-intercept is written as

$$z_\Phi = -\overline{MO}\tan\delta = -r\,\frac{2+\sqrt{2}\cos t}{2\sqrt{\left(\sqrt{2}+\cos t\right)^2-1}}\ , \tag{62}$$

and y-intercept y_Φ was already given by (4). Then by substituting the above interceptions in (60) we can get the following result:

$$g_1 = \frac{\dfrac{r^3\left(2+\sqrt{2}\cos t\right)^3}{8\sin t\ \cos t\ \sqrt{\left(\sqrt{2}+\cos t\right)^2-1}}}{\dfrac{r^2\left(2+\sqrt{2}\cos t\right)^3}{4\sqrt{2}\sin t\ \cos t\sqrt{\left(\sqrt{2}+\cos t\right)^2-1}}} = \frac{r}{\sqrt{2}}\ . \tag{63}$$

Accordingly, $g = g_1$, in other words the length of the perpendicular line from O to Φ is equal to the distance from O to line segment $\overline{AB}$. Hence line segment OS_d is perpendicular to Φ, therefore g is the height of the center of gravity O above Φ. Furthermore, g is not dependent upon t. For this reason, when the TCR wobbles on Φ, the height of the center of gravity is kept at a constant value of $r/\sqrt{2}$.

4. Moving Centrode M_C

The sphere with radius $r/\sqrt{2}$ and center O is denoted by G. From the results of the proceeding section 3, all generator lines $\overline{AB}$ contact with G at S_d, hence sphere G inscribes to Ω. (cf. Fig.7)
Let the coordinates of S_d be (x_{Sd}, y_{Sd}, z_{Sd}), then

$$x_{Sd} = \frac{r}{\sqrt{2}}\lambda_\phi\ ,\qquad y_{Sd} = \frac{r}{\sqrt{2}}\mu_\phi\ ,\qquad z_{Sd} = \frac{r}{\sqrt{2}}\nu_\phi\ , \tag{64}$$

where λ_ϕ、μ_ϕ、and ν_ϕ are the direction cosines of plane Φ and also line OS_d, therefore

$$\lambda_\phi = \frac{\frac{r}{\sqrt{2}}}{x_\phi}\ ,\qquad \mu_\phi = \frac{\frac{r}{\sqrt{2}}}{y_\phi}\ ,\qquad \nu_\phi = \frac{\frac{r}{\sqrt{2}}}{z_\phi}\ . \tag{65}$$

First substitute (61), (62) and the y-coordinate of (4) as y_ϕ in (65). Then by substituting the result of (65) in (64), we get

$$\left.\begin{aligned} x_{Sd} &= \frac{r\sin t}{2+\sqrt{2}\cos t} \\ y_{Sd} &= -\frac{r\cos t}{2+\sqrt{2}\cos t} \\ z_{Sd} &= \mp\frac{r\sqrt{\left(\sqrt{2}+\cos t\right)^2-1}}{2+\sqrt{2}\cos t} \end{aligned}\right\}\ . \tag{66}$$

Hence S_d can be described with Cartesian coordinate system O-xyz which is fixed to the TCR, and with the angular parameter t. In (66), the positive z_{Sd} is also regarded. Varying the parameter t, S_d moves on the surface of sphere G, and the locus of S_d is named M_C after the moving centrode. M_C is a spherical closed curve which is called a three-dimensional moving centrode of O in the theory of mechanisms. Where positive z_{Sd} indicates the upper half M_C and negative the lower half.

5. Fixed Centrode S_C

Wobbling the convex hull Ω with the inscribed sphere G on Φ, M_C is developed on Φ, and this curve is denoted by S_C, which is called a fixed centrode in the theory of mechanisms. This is the locus of the instantaneous center of O while the TCR is wobbling on Φ. Hence if we make M_C roll along S_C, the wobbling motion of the TCR reappears.
From Fig.3, the length of line segment $A_d S_d$ is equal to AS_d. Let L be this length, then from (3) and (66) we obtain

$$L = \sqrt{\left(\frac{r\sin t}{2+\sqrt{2}\cos t} - r\sin t\right)^2 + \left(-\frac{r\cos t}{2+\sqrt{2}\cos t} + r\,\frac{\sqrt{2}+2\cos t}{2}\right)^2 + \left(\pm\frac{r\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{2+\sqrt{2}\cos t}\right)^2}$$

$$= r\sqrt{1+\sqrt{2}\cos t}\ . \tag{67}$$

Hence if the X, Y coordinates of S_d on S_C are denoted as X_C and Y_C, as in (39) and (40), they can be written as

$$\left.\begin{aligned} X_C(t) &= X_A(t) + r\sqrt{1+\sqrt{2}\cos t}\,\cos\{\theta(t)+\beta(t)\} \\ Y_C(t) &= Y_A(t) + r\sqrt{1+\sqrt{2}\cos t}\,\sin\{\theta(t)+\beta(t)\} \end{aligned}\right\} . \tag{68}$$

(68) is a parameter representation of S_C by t. $(0 \le t \le \pi/2)$. The whole of S_C can be obtained, likewise C_{Ad} and C_{Bd}, with point reflection and line symmetries. However to connect the curve S_C smoothly, as shown in Fig.5, $S_d(\pi/2)$ must be midpoint of the line segment $A_d(\pi/2)\overline{B}_d(\pi/2)$. To meet the preceding requirement by using (8) and (67), let us calculate the ratio of L to line segment $A_d\overline{B}_d$ at $t = \pi/2$:

$$\left.\frac{\overline{A_d\overline{B}_d}}{L}\right)_{t=\frac{\pi}{2}} = \left.\frac{\dfrac{r\left(2+\sqrt{2}\cos t\right)}{\sqrt{1+\sqrt{2}\cos t}}}{r\sqrt{1+\sqrt{2}\cos t}}\right\}_{t=\frac{\pi}{2}} = \left.\frac{2+\sqrt{2}\cos t}{1+\sqrt{2}\cos t}\right)_{t=\frac{\pi}{2}} = 2\ . \tag{69}$$

By (69) the requirement that $S_d(\pi/2)$ must be at the midpoint of line segment $A_d(\pi/2)\overline{B}_d(\pi/2)$ is met. Hence point reflection of S_C can be achieved over $S_d(\pi/2)$. Accordingly line symmetry also becomes possible.

6. Illustration of the Development of the TCR; C_{Ad},C_{Bd} and Fixed Centrode S_C.

The curves shown in Fig.6 are obtained by the following procedure with load r = 1 and j = 50 (i.e., the range of t, from 0 to $\pi/2$, is equally divided into 50 parts) according to the program written with MATHEMATICA, using (37), (38), (39)a,(40)a, and (68) to compute the points, and then connecting them smoothly. Here $A_{ed}, \overline{A}_{ed}, B_{ed}$ and $\overline{B}_{ed}$ are cusps.

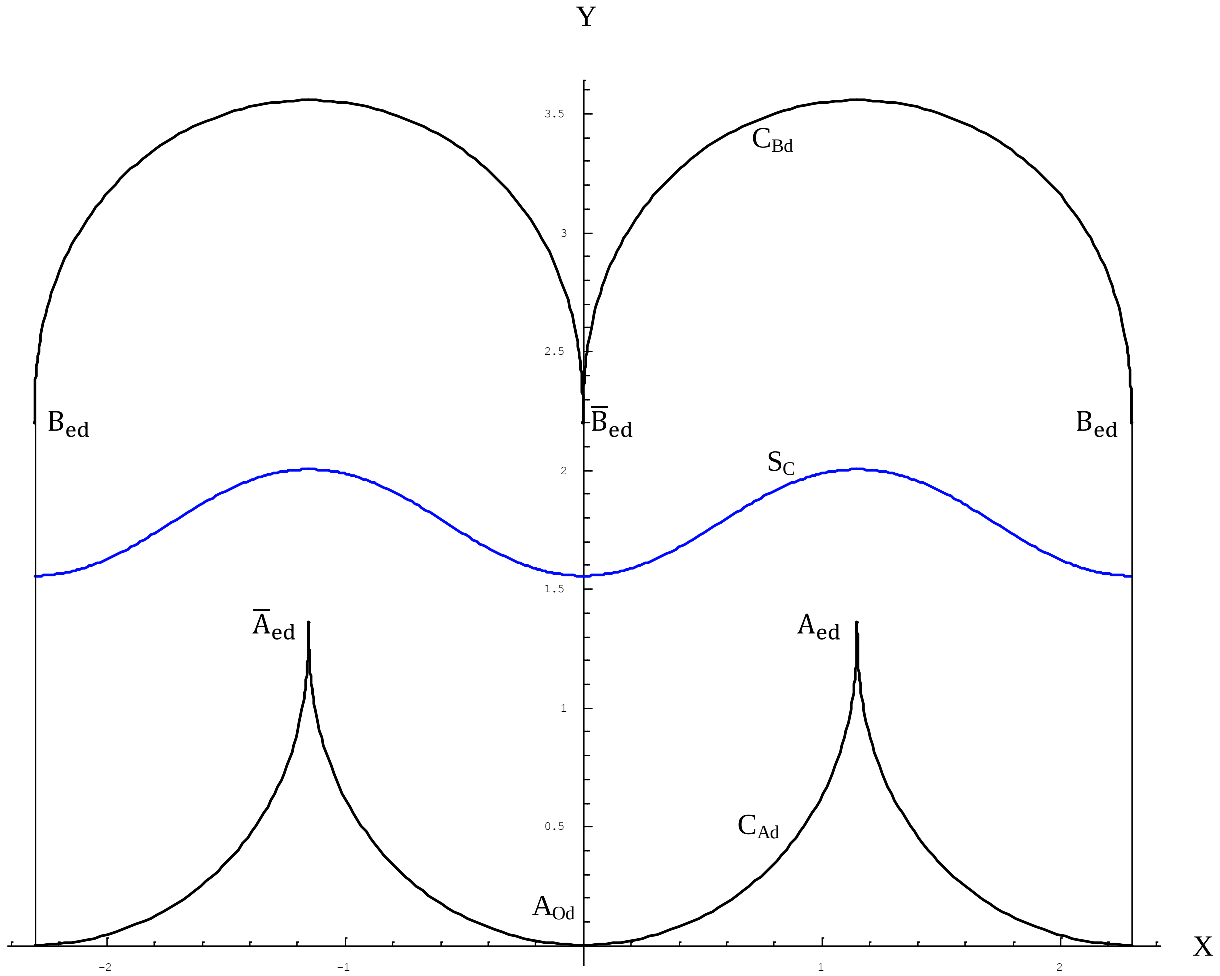

Fig.6 The Development of the TCR; C_{Ad}, C_{Bd} and Fixed Centrode S_C

7. Convex Hull Ω with Generator Lines $A\overline{B}$ (AB), Sphere G, and Moving Centrode M_C.

Let us introduce τ to describe the coordinates of B on C_B and A on C_A, then from Fig.1 we obtain

$$B = \left\{0\,,\quad r\left(\frac{\sqrt{2}+2\cos\tau}{2}\right),\quad r\sin\tau\right\} \tag{70}$$

$$A = \left\{\pm\frac{r\sqrt{\left(\sqrt{2}+\cos\tau\right)^2-1}}{1+\sqrt{2}\cos\tau}\,,\qquad -\frac{r\sqrt{2}}{2\left(1+\sqrt{2}\cos\tau\right)}\,,\qquad 0\right\}\ . \tag{71}$$

We write both of negative and the positive x-coordinates down in (71). To calculate the half circles of C_A and C_B discretely, we divide the range of parameter t and τ $(-\pi/2$ to $\pi/2)$ equally. The figures shown in Fig.7 are described by (3), (6), (66), (70), (71), and inscribed sphere G; $x^2+y^2+z^2 = r^2/2$.

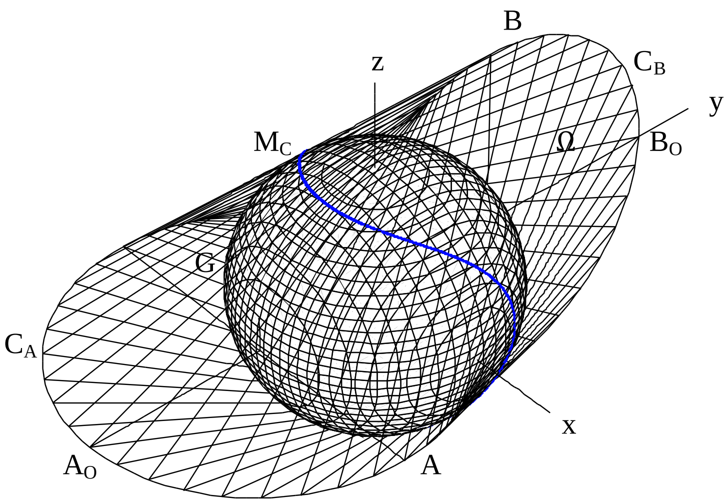

Fig.7: Convex Hull Ω composed of the generators, inscribed sphere G, and M_C.

8. Vector Equation of Convex Hull Ω

We want to obtain the vector equation of Ω which is described with Cartesian coordinate system O-xyz. The vector $\mathbf{Q_A}$, and absolute value of $\mathbf{Q_A}$, is obtained with (43) and (8) respectively.
Hence the unit-vector $\vec{e}_{AB}$ of $\mathbf{Q_A}$ can be written as

$$\vec{e}_{AB}(t) = \frac{\mathbf{Q_A}}{\overline{AB}} = \left(e_{ABx}\,, e_{ABy}\,,\ e_{ABz}\right)$$

$$= \left\{-\frac{\sin t\sqrt{1+\sqrt{2}\,\cos t}}{2+\sqrt{2}\cos t}\,,\quad \frac{1+\sqrt{2}\cos t+\cos^2 t}{\left(\sqrt{2}+\cos t\right)\sqrt{1+\sqrt{2}\cos t}}\,,\quad \pm\frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2-1}}{\sqrt{2}\left(\sqrt{2}+\cos t\right)\sqrt{1+\sqrt{2}\cos t}}\right\}, \tag{72}$$

where the range of t is $-(\pi/2+\delta_O) \le t \le (\pi/2+\delta_O)$, and δ_O is given by substituting t = 0 in (5)

$$\delta_O = \sin^{-1}\left(\frac{1}{1+\sqrt{2}}\right)\ . \tag{73}$$

Let an arbitrary point on the generator of Ω be $P_r(x_r, y_r, z_r)$, with position vector $\mathbf{P_r}$. Then

$$\mathbf{P_r}(t\,,u) = \mathbf{A}(t) + u\,\vec{e}_{AB}(t) \qquad\qquad 0 \le u \le \overline{AB}\quad , \tag{74}$$

where $\mathbf{A}(t)$ is the position vector of A, described with O-xyz coordinates, and given by (3), u is a scalar with which the distance from A to P_r along the generator line is specified, and corresponding to each A, u varies from 0 to the absolute value of $\mathbf{Q_A}$ (= $\overline{AB}$). When u reaches a maximum value, P_r becomes B on C_B. (74) is the parametric vector equation of ruled surface Ω with parameters t and u, however the maximum value of u is specified by t. Now as in section 7, to obtain a smooth illustration of Ω, the range of parameter t and τ $(-\pi/2 \sim \pi/2)$ is divided equally, and the half circles of C_A and C_B are calculated discretely. First we stipulate that the vector $\mathbf{Q_B}$ is the same as vector $\mathbf{Q_A}$ but in the opposite direction, i.e. the direction of generator BA is the opposite of AB. From (70) and (71), $\mathbf{Q_B}$ can be obtained directly as

$$\mathbf{Q}_{\mathrm{B}} = \left\{ \pm\frac{r\sqrt{\left(\sqrt{2}+\cos\tau\right)^2-1}}{1+\sqrt{2}\cos\tau}\ , \quad -\frac{r\sqrt{2}\left(1+\sqrt{2}\cos\tau+\cos^2\tau\right)}{1+\sqrt{2}\cos\tau}\ , \quad -r\sin\tau \right\} . \tag{75}$$

The absolute values of $\mathbf{Q_B}$ and $\mathbf{Q_A}$ are the same, therefore the unit-vector $\vec{e}_{BA}$ can be obtained as

$$\vec{e}_{BA} = \frac{\mathbf{Q}_B}{\overline{AB}(\tau)} = \left(e_{BAx}\,,\, e_{BAy}\,,\, e_{BAz}\right)$$

$$= \left\{ \pm\frac{\sqrt{\left(\sqrt{2}+\cos\tau\right)^2-1}}{\left(2+\sqrt{2}\cos\tau\right)\sqrt{1+\sqrt{2}\cos\tau}}\ ,-\frac{1+\sqrt{2}\cos\tau+\cos^2\tau}{\left(\sqrt{2}+\cos\tau\right)\sqrt{1+\sqrt{2}\cos\tau}}\ ,-\frac{\sin\tau\sqrt{1+\sqrt{2}\cos\tau}}{2+\sqrt{2}\cos\tau} \right\} . \tag{76}$$

Accordingly, the position vector $\mathbf{P_{rb}}$ corresponds to an arbitrary point on generator BA and can be written as

$$\mathbf{P_{rb}}(\tau\,,u_b) = \mathbf{B}(\tau) + u_b\,\vec{e}_{BA}(\tau) \qquad 0 \le u_b \le \overline{AB}\ , \tag{77}$$

where $\mathbf{B}(\tau)$ is the position vector of B, described with O-xyz coordinates, and given by (70). u_b is a scalar with which the distance from B to P_{rb} along generator line BA is specified. Corresponding to each B, u_b varies from 0 to the absolute value of $\mathbf{Q_B}$ ($= \overline{BA}$). When u_b reaches the maximum value, P_{rb} becomes A on C_A. (77) is parametric vector equation of ruled surface Ω with parameters τ and u_b. However the maximum value of u_b is specified by τ.
The components of (74) and (77) can be described as

$$\left.\begin{aligned}
x_r(t\,,u) &= r\sin t - u\,\frac{\sin t\sqrt{1+\sqrt{2}\,\cos t}}{2+\sqrt{2}\cos t}\\
y_r(t\,,u) &= -r\left(\frac{\sqrt{2}}{2}+\cos\,t\right) + u\,\frac{1+\sqrt{2}\cos t+\cos^2 t}{\left(\sqrt{2}+\cos t\right)\sqrt{1+\sqrt{2}\cos t}}\\
z_r(t\,,u) &= 0 \pm u\,\frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2-1}}{\sqrt{2}\left(\sqrt{2}+\cos t\right)\sqrt{1+\sqrt{2}\cos t}}\\
&0 \le t \le \frac{\pi}{2}\ , \qquad 0 \le u \le \overline{AB}\,(t)
\end{aligned}\right\} \tag{78}$$

$$\left.\begin{aligned}
x_{rb}(\tau\,,u_b) &= 0 \pm u_b\,\frac{\sqrt{\left(\sqrt{2}+\cos\tau\right)^2-1}}{\left(2+\sqrt{2}\cos\tau\right)\sqrt{1+\sqrt{2}\cos\tau}}\\
y_{rb}(\tau\,,u_b) &= r\left(\frac{\sqrt{2}}{2}+\cos\,\tau\right) - u_b\,\frac{1+\sqrt{2}\cos\tau+\cos^2\tau}{\left(\sqrt{2}+\cos\tau\right)\sqrt{1+\sqrt{2}\cos\tau}}\\
z_{rb}(\tau\,,u_b) &= r\sin\tau - u_b\,\frac{\sin\tau\sqrt{1+\sqrt{2}\cos\tau}}{2+\sqrt{2}\cos\tau}\\
&0 \le \tau \le \frac{\pi}{2}\ , \qquad 0 \le u_b \le \overline{AB}\,(\tau)
\end{aligned}\right\} . \tag{79}$$

Now $z_r(t,u)$ of (78) and $x_{rb}(\tau,u_b)$ of (79) have both positive and negative values. Then the convex hull Ω shown in Fig.8 is plotted with the four parts together.

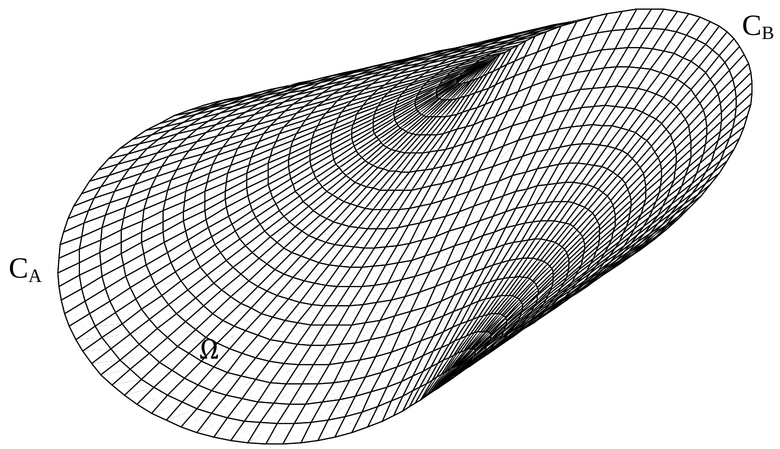

Fig.8: Convex hull Ω of the TCR by vector equation

9. Wobbling Posture of Convex Hull Ω on Φ (XY- plane)

In order to describe the wobbling posture of Ω, we introduce the moving frame of Ω with origin A_d on C_{Ad} for $-(\pi/2+\delta_O) \le t \le (\pi/2+\delta_O)$. (cf. Fig.9)

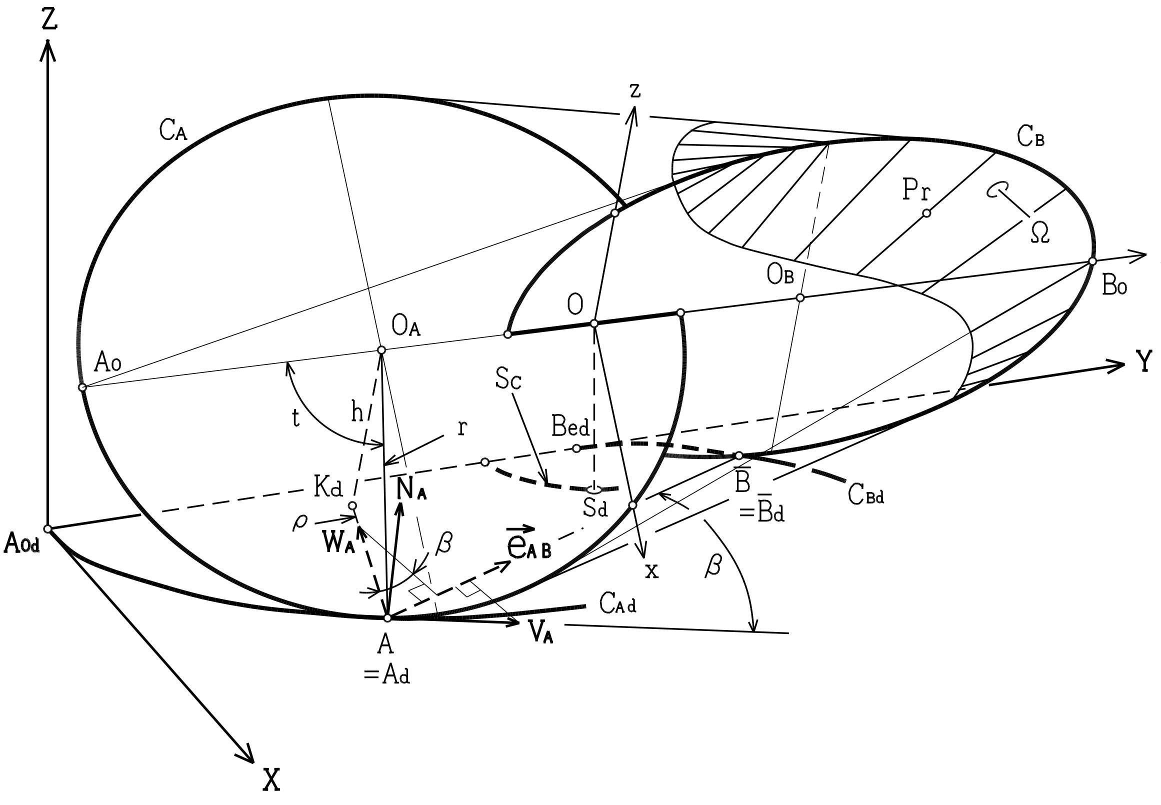

Fig.9: Wobbling posture of convex hull Ω on Φ

C_A contacts XY-plane (Φ) at A, and corresponds to A_d on C_{Ad}. $\mathbf{V_A}$ is defined as the unit tangent vector to C_{Ad} at A_d, as shown in Fig.9. The unit vector $\mathbf{W_A}$, which is specified in Φ and pointing to K_d, is orthogonal to $\mathbf{V_A}$ at A_d. Further the unit vector $\mathbf{N_A}$ is defined as the cross product of $\mathbf{V_A}$ and $\mathbf{W_A}$, i.e. $\mathbf{N_A}$ is perpendicular to Φ. Now Cartesian frame (A_d: $\mathbf{V_A}$, $\mathbf{W_A}$, $\mathbf{N_A}$) is completed and will follow A_d, which moves on C_{Ad} while Ω is wobbling on XY-plane (Φ). The Cartesian frame (A_d: $\mathbf{V_A}$, $\mathbf{W_A}$, $\mathbf{N_A}$) can be described with both O-xyz coordinate systems fixed to Ω and A_{Od}-XYZ to Φ.

Hence if we take a point on the surface of Ω, and denote $P_r(\zeta, \psi, \omega)$ of this point with Cartesian moving frame (A_d: $\mathbf{V_A}$, $\mathbf{W_A}$, $\mathbf{N_A}$), then the following equations are obtained:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \\ 0 \end{pmatrix} + \begin{pmatrix} x_{VA} & x_{WA} & x_{NA} \\ y_{VA} & y_{WA} & y_{NA} \\ z_{VA} & z_{WA} & z_{NA} \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta} \\ \psi \\ \omega \end{pmatrix} \tag{80}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X_A \\ Y_A \\ 0 \end{pmatrix} + \begin{pmatrix} X_{VA} & X_{WA} & X_{NA} \\ Y_{VA} & Y_{WA} & Y_{NA} \\ Z_{VA} & Z_{WA} & Z_{NA} \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta} \\ \psi \\ \omega \end{pmatrix} . \tag{81}$$

The first terms of the right-hand side of (80) are the coordinates of A written with O-xyz coordinate systems, then it can be obtained from (54), and the vector matrix of the second term can be got from the elements of ($\mathbf{V_A}$, $\mathbf{W_A}$, $\mathbf{N_A}$). Similarly, the first terms of the right-hand side of (81) are the coordinates of A_d described with A_{Od}-XYZ coordinate systems, then it can be obtained from (37) and (38), and the vector matrix of the second term can be got from the elements of $(\mathbf{V_{Ad}}, \mathbf{W_{Ad}}, \mathbf{N_{Ad}})$ which are obtained by describing the elements of ($\mathbf{V_A}$, $\mathbf{W_A}$, $\mathbf{N_A}$) with A_{Od}-XYZ coordinate systems. By eliminating (ζ, ψ, ω) from (80) and (81) we can obtain the moving frame, with which the rule of transformation between the O-xyz coordinate systems and A_{Od}-XYZ is described, corresponding to Ω wobbling on Φ, where the elements of $\mathbf{V_A}$ are given by (42).

In Fig.9, by using the relation of $\vec{e}_{AB} = \mathbf{V_A}\cos\beta + \mathbf{W_A}\sin\beta$, $\mathbf{W_A}$ is described as

$$\mathbf{W_A} = \frac{\vec{e}_{AB} - \mathbf{V_A}\cos\beta}{\sin\beta} . \tag{82}$$

Hence by substituting (42), (45), (46), and (72) in (82), the elements of $\mathbf{W_A}$ can be written as

$$\begin{aligned} \mathbf{W_A} &= (x_{WA}, y_{WA}, z_{WA}) \\ &= \left\{ -\frac{\sin t\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{\sqrt{2}+\cos t}, \quad \frac{\cos t\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{\sqrt{2}+\cos t}, \quad -\frac{1}{\sqrt{2}+\cos t} \right\} . \end{aligned} \tag{83}$$

Now the sign of z_{WA} can be both positive and negative in the range of $-(\pi/2 + \delta_O) \le t \le (\pi/2 + \delta_O)$, however if the TCR shown in Fig.2 and 3 touches Φ at a negative z-coordinate, then we take negative z_{WA}. By definition the unit vector $\mathbf{N_A}$ can be written as

$$\begin{aligned} \mathbf{N_A} &= \mathbf{V_A} \times \mathbf{W_A} = (x_{NA}, y_{NA}, z_{NA}) \\ &= \left\{ -\frac{\sin t}{\sqrt{2}+\cos t}, \quad \frac{\cos t}{\sqrt{2}+\cos t}, \quad \frac{\sqrt{\left(\sqrt{2}+\cos t\right)^2 - 1}}{\sqrt{2}+\cos t} \right\} . \end{aligned} \tag{84}$$

Hence all the elements of the vectors ($\mathbf{V_A}$, $\mathbf{W_A}$, $\mathbf{N_A}$) are obtained by (42), (83), and (84). They are described with O-xyz coordinate systems.

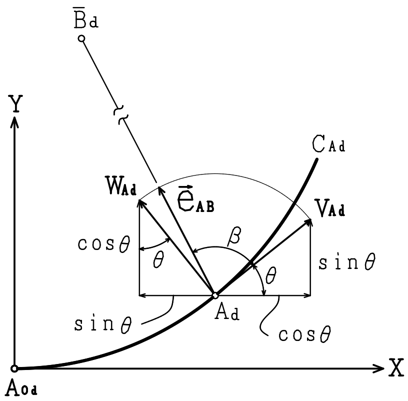

Fig.10

Let us consider describing the elements of $(\mathbf{V_{Ad}}, \mathbf{W_{Ad}}, \mathbf{N_{Ad}})$ with A_{Od}-XYZ coordinate systems.

Where Fig.10 is the upper view of Fig.9 from the positive Z-axis, seeing Fig.10, and regarding that $\vec{e}_{AB}$ is a unit vector and the unit vector $\mathbf{N_A}$ has only a Z-element, we can describe all the elements of $(\mathbf{V_{Ad}}, \mathbf{W_{Ad}}, \mathbf{N_{Ad}})$ with A_{Od}-XYZ coordinate systems as below:

$$\left.\begin{aligned}\mathbf{V}_{\mathbf{Ad}} &= (X_{VA}\ ,Y_{VA}\ ,\ Z_{VA}) = \{\cos\theta(t)\ ,\ \sin\theta(t)\ ,0\}\\ \mathbf{W}_{\mathbf{Ad}} &= (X_{WA}\ ,Y_{WA}\ ,Z_{WA}) = \{-\sin\theta(t)\ ,\ \cos\theta(t)\ ,0\}\\ \mathbf{N}_{\mathbf{Ad}} &= (X_{NA}\ ,Y_{NA}\ ,\ Z_{NA}) = (0\ ,\ \ 0\ ,\ \ 1)\end{aligned}\right\} \quad . \tag{85}$$

Now by shifting the column vector of the first term of the right-hand side of (80) to the left-hand side we get

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix} - \begin{pmatrix} x_A\\ y_A\\ 0\end{pmatrix} = \begin{pmatrix} x_{VA} & x_{WA} & x_{NA}\\ y_{VA} & y_{WA} & y_{NA}\\ z_{VA} & z_{WA} & z_{NA}\end{pmatrix}\begin{pmatrix} \boldsymbol{\zeta}\\ \psi\\ \omega\end{pmatrix} \quad . \tag{86}$$

Liner-algebra leads to

$$\begin{pmatrix} \boldsymbol{\zeta}\\ \psi\\ \omega\end{pmatrix} = \frac{1}{\begin{vmatrix} x_{VA} & x_{WA} & x_{NA}\\ y_{VA} & y_{WA} & y_{NA}\\ z_{VA} & z_{WA} & z_{NA}\end{vmatrix}}\begin{pmatrix} \begin{vmatrix} x & x_{WA} & x_{NA}\\ y & y_{WA} & y_{NA}\\ z & z_{WA} & z_{NA}\end{vmatrix} - \begin{vmatrix} x_A & x_{WA} & x_{NA}\\ y_A & y_{WA} & y_{NA}\\ 0 & z_{WA} & z_{NA}\end{vmatrix}\\ \begin{vmatrix} x_{VA} & x & x_{NA}\\ y_{VA} & y & y_{NA}\\ z_{VA} & z & z_{NA}\end{vmatrix} - \begin{vmatrix} x_{VA} & x_A & x_{NA}\\ y_{VA} & y_A & y_{NA}\\ z_{VA} & 0 & z_{NA}\end{vmatrix}\\ \begin{vmatrix} x_{VA} & x_{WA} & x\\ y_{VA} & y_{WA} & y\\ z_{VA} & z_{WA} & z\end{vmatrix} - \begin{vmatrix} x_{VA} & x_{WA} & x_A\\ y_{VA} & y_{WA} & y_A\\ z_{VA} & z_{WA} & 0\end{vmatrix}\end{pmatrix} \quad . \tag{87}$$

And calculating the denominator of the right-hand side of (87) with (42), (83), and (84) we get

$$\begin{vmatrix} x_{VA} & x_{WA} & x_{NA}\\ y_{VA} & y_{WA} & y_{NA}\\ z_{VA} & z_{WA} & z_{NA}\end{vmatrix} = 1 \quad . \tag{88}$$

By using (85) we can describe the vector matrix $(\mathbf{V}_{\mathbf{Ad}}\ ,\mathbf{W}_{\mathbf{Ad}}\ ,\mathbf{N}_{\mathbf{Ad}})$, which is the second term on right-hand side of (81), as below:

$$\begin{pmatrix} X_{VA} & X_{WA} & X_{NA}\\ Y_{VA} & Y_{WA} & Y_{NA}\\ Z_{VA} & Z_{WA} & Z_{NA}\end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} \quad . \tag{89}$$

Therefore, by substituting (87) in (81), we can obtain following result:

$$\begin{aligned}\begin{pmatrix} X\\ Y\\ Z\end{pmatrix} = \begin{pmatrix} X_A\\ Y_A\\ 0\end{pmatrix} - \begin{pmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \begin{vmatrix} x_A & x_{WA} & x_{NA}\\ y_A & y_{WA} & y_{NA}\\ 0 & z_{WA} & z_{NA}\end{vmatrix}\\ \begin{vmatrix} x_{VA} & x_A & x_{NA}\\ y_{VA} & y_A & y_{NA}\\ z_{VA} & 0 & z_{NA}\end{vmatrix}\\ \begin{vmatrix} x_{VA} & x_{WA} & x_A\\ y_{VA} & y_{WA} & y_A\\ z_{VA} & z_{WA} & 0\end{vmatrix}\end{pmatrix}\\ + \begin{pmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} \begin{vmatrix} x & x_{WA} & x_{NA}\\ y & y_{WA} & y_{NA}\\ z & z_{WA} & z_{NA}\end{vmatrix}\\ \begin{vmatrix} x_{VA} & x & x_{NA}\\ y_{VA} & y & y_{NA}\\ z_{VA} & z & z_{NA}\end{vmatrix}\\ \begin{vmatrix} x_{VA} & x_{WA} & x\\ y_{VA} & y_{WA} & y\\ z_{VA} & z_{WA} & z\end{vmatrix}\end{pmatrix} \quad .\end{aligned} \tag{90}$$

To simplify we introduce $S = \sin t\,,\ \ C = \cos t\,,\ \ -\left(\frac{\pi}{2} + \delta_O\right) \le t \le \left(\frac{\pi}{2} + \delta_O\right)\,,\ \ S^2 + C^2 = 1$, where δ_O is given by (73). By substituting the elements of (42), (83), and (84) in (90), computing the value of determinants, and calculating the matrices, we obtain the following matrix representation:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X_A \\ Y_A \\ 0 \end{pmatrix} + \frac{r\sqrt{2}}{2}\begin{pmatrix} S\ \cos\theta - \sin\theta\sqrt{\left(\sqrt{2}+C\right)^2-1} \\ S\ \sin\theta + \cos\theta\sqrt{\left(\sqrt{2}+C\right)^2-1} \\ 1 \end{pmatrix}$$

$$+\begin{pmatrix} C\ \cos\theta + \dfrac{S\ \sin\theta\sqrt{\left(\sqrt{2}+C\right)^2-1}}{\sqrt{2}+C}, & S\ \cos\theta - \dfrac{C\ \sin\theta\sqrt{\left(\sqrt{2}+C\right)^2-1}}{\sqrt{2}+C}, & \dfrac{\sin\theta}{\sqrt{2}+C} \\ C\ \sin\theta - \dfrac{S\ \cos\theta\sqrt{\left(\sqrt{2}+C\right)^2-1}}{\sqrt{2}+C}, & S\ \sin\theta + \dfrac{C\ \cos\theta\sqrt{\left(\sqrt{2}+C\right)^2-1}}{\sqrt{2}+C}, & -\dfrac{\cos\theta}{\sqrt{2}+C} \\ -\dfrac{S}{\sqrt{2}+C}, & \dfrac{C}{\sqrt{2}+C}, & \dfrac{\sqrt{\left(\sqrt{2}+C\right)^2-1}}{\sqrt{2}+C} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (91)$$

The matrix equation (91) is the moving frame of Ω with origin A_d on C_{Ad}.
By substituting x = y = z = 0 in (91) we get the space curve which is the locus of O while Ω wobbles on XY-plane (Φ). From this result we can find that the value of Z equals $r/\sqrt{2}$, which is the same result as (63), and is independent of t. If we ignore Z, then we get a two-dimensional curve, which is equal to S_C given by (68), and S_C is an orthogonal projection of the locus of O onto Φ. We can prove this fact by substituting (45) and (46) in the expanded result of (68) with the addition theorem for trigonometric functions. The third term of (91) is the inner product of the matrix and column vectors.

The wobbling posture of Ω on Φ corresponding to $t = 0.7\left\{\frac{\pi}{2} + \sin^{-1}\left(\frac{1}{1+\sqrt{2}}\right)\right\}$ is obtained by substituting the above mentioned t and the equations of Ω into (91), where the equations of Ω are described with O-xyz coordinate systems and represented by parameter t. The wobbling posture of Ω is illustrated in Fig.11 with development curves C_{Ad}, C_{Bd}, fixed centrode S_C, moving centrode M_C, inscribed sphere G, and locus of center of gravity O (red curve).

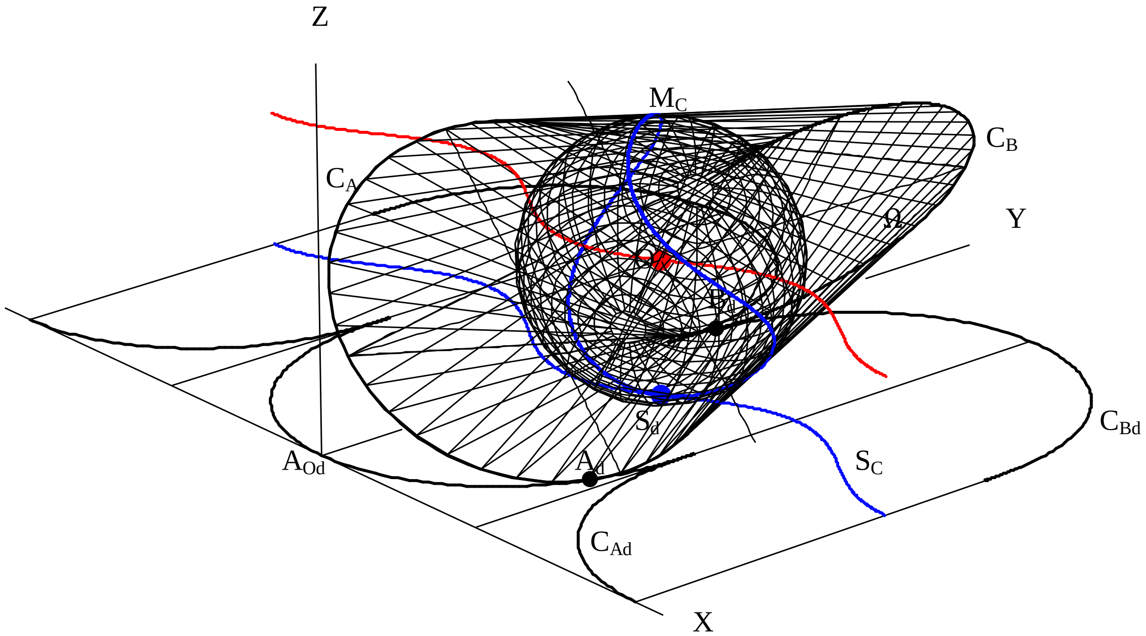

Fig.11: Wobbling posture of convex hull Ω of the TCR on the development curves

10. Surface Area S_Ω and Volume V_Ω of Convex Hull Ω

S_Ω equals the area under the development curves C_{Ad}, C_{Bd} shown in Fig.6. Hence it can be given by increasing the area which is surrounded by two lines $\overline{A_{Od}\,\overline{B}_{ed}}$, $\overline{A_d(\pi/2)\overline{B}_d(\pi/2)}$ and two curves $\widehat{A_{Od}A_d}(\pi/2)$, $\widehat{\overline{B}_{ed}\overline{B}_d}(\pi/2)$ in Fig.5 by eight times. The movements of A_d on C_{Ad} and $\overline{B}_d$ on C_{Bd} during the very short time interval dT are denoted by (dX_A, dY_A) and (dX_B, dY_B) along the X- and Y-axis respectively. Then the velocity vectors $\mathbf{U_{Ad}}$ and $\mathbf{U_{Bd}}$ can be written as

$$\mathbf{U_{Ad}} = \left\{\frac{dX_A}{dT}\ , \qquad \frac{dY_A}{dT}\right\} \tag{92}$$

$$\mathbf{U_{Bd}} = \left\{\frac{dX_B}{dT}\ , \qquad \frac{dY_B}{dT}\right\} \quad . \tag{93}$$

Let dt be the small increment of angular parameter t during time interval dT. Then the angular velocity ω_A, which is defined as the rotating speed of radius O_AO around O, can be obtained as

$$\omega_A = \frac{dt}{dT} \quad . \tag{94}$$

There is no problem to assume $\omega_A = 1$ for the purpose of obtaining the very small area dS which is swept by line segment $A_d(t)\overline{B}_d(t)$ during dT, therefore, from (94)

$$dT = dt \quad . \tag{95}$$

By substituting (95) in (92) and (93), time derivatives can be changed into angular derivatives as

$$\mathbf{U_{Ad}} = \left\{\frac{dX_A}{dt}\ , \qquad \frac{dY_A}{dt}\right\} \tag{96}$$

$$\mathbf{U_{Bd}} = \left\{\frac{dX_B}{dt}\ , \qquad \frac{dY_B}{dt}\right\} \quad . \tag{97}$$

Further, by substituting (49) in (96), we obtain directly

$$\mathbf{U_{Ad}} = \{r\cos\theta(t)\ , \qquad r\sin\theta(t)\} \quad . \tag{96)a}$$

By using the denominator and numerator of (48) we can calculate (97) as below:

$$\mathbf{U_{Bd}} = \left\{\frac{dX_A(t)}{dt} + \frac{d\left(\overline{A_d\overline{B}_d}\right)}{dt}\cos\{\theta(t)+\beta(t)\} - \left(\overline{A_d\overline{B}_d}\right)\sin\{\theta(t)+\beta(t)\}\left\{\frac{d\theta(t)}{dt} + \frac{d\beta(t)}{dt}\right\}\ ,\right.$$
$$\left.\frac{dY_A(t)}{dt} + \frac{d\left(\overline{A_d\overline{B}_d}\right)}{dt}\sin\{\theta(t)+\beta(t)\} + \left(\overline{A_d\overline{B}_d}\right)\cos\{\theta(t)+\beta(t)\}\left\{\frac{d\theta(t)}{dt} + \frac{d\beta(t)}{dt}\right\}\right\} . \tag{97)a}$$

By substituting (8), (36), (45), (46), and from (49) to (52) in (97)a, we can obtain $\mathbf{U_{Bd}}$. If we consider that the ends of line segment $A_d(t)\overline{B}_d(t)$ have velocity vectors $\mathbf{U_{Ad}}$ and $\mathbf{U_{Bd}}$ as shown in Fig.12, then dS can be obtained as a cross product of $\mathbf{Q_A}$ and the mean vector of $\mathbf{U_{Ad}}$ and $\mathbf{U_{Bd}}$.
That is

$$dS = \left|\frac{\mathbf{U_{Ad}} + \mathbf{U_{Bd}}}{2} \times \mathbf{Q_A}\right| dT \quad . \tag{98}$$

The procedure with which (98) is derived is noted below.

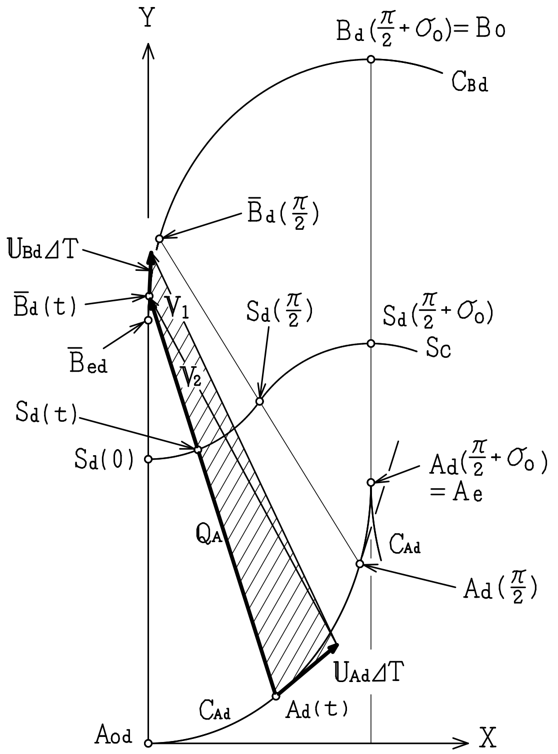

Fig.12

In Fig.12, the vector $\mathbf{Q_A}$ is defined by the line segment $A_d\bar{B}_d$. Let $\mathbf{U_{Ad}}$ and $\mathbf{U_{Bd}}$ be velocity vectors of both ends of line segment $A_d\bar{B}_d$. Hence $\mathbf{U_{Ad}}$dT and $\mathbf{U_{Bd}}$dT are the small displacement vectors of the ends of line segment $A_d\bar{B}_d$ during the very short time interval ΔT.
The vector from the target of $\mathbf{U_{Ad}}$dT to the target of $\mathbf{U_{Bd}}$dT is denoted by $\mathbf{V_1}$, and the vector from the target of $\mathbf{U_{Ad}}$dT to the source of $\mathbf{U_{Bd}}$dT is denoted by $\mathbf{V_2}$.
The very small quadrilateral area, which is enclosed by four vectors, $\mathbf{Q_A}$, $\mathbf{U_{Ad}}$dT, $\mathbf{U_{Bd}}$dT, and $\mathbf{V_1}$, and shown as the area shadowed by hatching, is denoted as ΔS, where from Fig.12

$$\mathbf{V_2} = \mathbf{Q_A} - \mathbf{U_{Ad}}\Delta T \quad , \tag{99}$$

and

$$\mathbf{V_1} = \mathbf{V_2} + \mathbf{U_{Bd}}\Delta T = \mathbf{Q_A} - \mathbf{U_{Ad}}\Delta T + \mathbf{U_{Bd}}\Delta T \ . \tag{100}$$

Because the intersection of $\mathbf{V_1}$ and $\mathbf{Q_A}$ does not exist on line segment $A_d\bar{B}_d$ (but both ends are included), ΔS can be described with a cross product.

Then

$$\begin{aligned}
\Delta S &= \left|\frac{\mathbf{U_{Ad}}\Delta T \times \mathbf{Q_A} + \mathbf{V_1} \times \mathbf{V_2}}{2}\right| \\
&= \left|\frac{\mathbf{U_{Ad}}\Delta T \times \mathbf{Q_A} + (\mathbf{Q_A} - \mathbf{U_{Ad}}\Delta T + \mathbf{U_{Bd}}\Delta T) \times (\mathbf{Q_A} - \mathbf{U_{Ad}}\Delta T)}{2}\right| \\
&= \left|\frac{\mathbf{U_{Ad}}\Delta T \times \mathbf{Q_A} + \{(\mathbf{Q_A} - \mathbf{U_{Ad}}\Delta T) \times (\mathbf{Q_A} - \mathbf{U_{Ad}}\Delta T) + \mathbf{U_{Bd}}\Delta T \times (\mathbf{Q_A} - \mathbf{U_{Ad}}\Delta T)\}}{2}\right| \\
&= \left|\frac{\mathbf{U_{Ad}}\Delta T \times \mathbf{Q_A} + \mathbf{U_{Bd}}\Delta T \times \mathbf{Q_A} - (\mathbf{U_{Bd}} \times \mathbf{U_{Ad}})\Delta T^2}{2}\right| \quad .
\end{aligned} \tag{101}$$

We can ignore the third term of the numerator of (101) because it is negligibly small. Therefore when ΔT approaches zero, the limit of (101) becomes (98). Further, by regarding (95), we can get

$$dS = \left|\frac{\mathbf{U_{Ad}} + \mathbf{U_{Bd}}}{2} \times \mathbf{Q_A}\right| dt \quad . \tag{102}$$

However $\mathbf{Q_A}$ must be described with A_{Od}-XYZ coordinate systems. That is, from Fig.5 and (8)

$$\mathbf{Q}_A = \left\{ r\frac{\sqrt{2}\left(\sqrt{2}+\cos t\right)}{\sqrt{1+\sqrt{2}\cos t}}\cos\{\theta(t)+\beta(t)\} \ , \qquad r\frac{\sqrt{2}\left(\sqrt{2}+\cos t\right)}{\sqrt{1+\sqrt{2}\cos t}}\sin\{\theta(t)+\beta(t)\} \right\} \ . \tag{103}$$

By substituting (103), (96)a, and (97)a into (102), the surface area S_Ω of convex hull Ω is obtained as

$$\begin{aligned}
S_\Omega &= 8\int_0^{\frac{\pi}{2}} \left|\frac{\mathbf{U}_{Ad} + \mathbf{U}_{Bd}}{2} \times \mathbf{Q}_A\right| dt \\
&= 8\, r^2 \int_0^{\frac{\pi}{2}} \left| \left(2+\sqrt{2}\cos t\right) \left\{ \frac{\sin\{\theta(t)+\beta(t)\}\cos\theta(t) - \cos\{\theta(t)+\beta(t)\}\sin\theta(t)}{\sqrt{1+\sqrt{2}\cos t}} \right.\right. \\
&\qquad \left.\left. - \frac{\sqrt{2}\left(4+9\sqrt{2}\cos t+14\cos^2 t+3\sqrt{2}\cos^3 t\right)}{4\left(1+\sqrt{2}\cos t\right)^2\sqrt{1+2\sqrt{2}\cos t+\cos^2 t}} \right\} \right| dt \ .
\end{aligned}$$

By applying the addition theorem for trigonometric functions to the above equation, it can be written as

$$= 8\,\mathrm{r}^2 \int_0^{\frac{\pi}{2}} \left| \left(2+\sqrt{2}\cos t\right) \left\{ \frac{\sin\beta(t)}{\sqrt{1+\sqrt{2}\cos t}} - \frac{\sqrt{2}\left(4+9\sqrt{2}\cos t+14\cos^2 t+3\sqrt{2}\cos^3 t\right)}{4\left(1+\sqrt{2}\cos t\right)^2 \sqrt{\left(\sqrt{2}+\cos t\right)^2-1}} \right\} \right| \,\mathrm{dt}\,.$$

Moreover, substituting (46) for $\sin\beta(t)$ in the above equation, we get

$$= 8\,\mathrm{r}^2 \int_0^{\frac{\pi}{2}} \left| -\frac{\left(2+\sqrt{2}\cos t\right)^2\left(1+\sqrt{2}\cos t+\cos^2 t\right)}{2\sqrt{2}\left(1+\sqrt{2}\cos t\right)^2\sqrt{\left(\sqrt{2}+\cos t\right)^2-1}} \right| \,\mathrm{dt} \quad . \tag{104)a}$$

The above integration yields

$$\mathrm{S}_\Omega = 8\,\mathrm{r}^2 \left\{ \frac{\sqrt{2}-1}{2} + \mathrm{E}[\varphi_s, k_s] + \sqrt{2}\left(\Pi[-a, \varphi_s, k_s] + \Pi[b, \varphi_s, k_s] - \mathrm{F}[\varphi_s, k_s] \right) \right\} \quad , \tag{104}$$

where $\mathrm{E}[\varphi, k]$ is the elliptic integral of the second kind of Legendre-Jacobi's standard form. That is

$$\mathrm{E}[\varphi, k] = \int_0^{\sin\varphi} \sqrt{\frac{1-k\,t^2}{1-t^2}}\,\mathrm{dt} \tag{104)b}$$

and, from (34) and (35), the elliptic integral of the first and third kinds has been written as $\mathrm{F}[\varphi, k]$ and $\Pi[n, \varphi, k]$ respectively . Now, a and b are defined by (29), further $k_s = -1$, and $\varphi_s = \sin^{-1}\sqrt{\sqrt{2}-1}$.
Hence the numerical result of (104) becomes

$$\mathrm{S}_\Omega = 8\,\mathrm{r}^2(1.74044761\cdots) = 13.92358088\cdots \mathrm{r}^2 \quad . \tag{105}$$

The surface area of a sphere of radius r is $4\pi r^2 = 12.56637061\cdots r^2$. Accordingly, S_Ω is about 10.80 % wider compared with the surface area of a sphere of same radius as the TCR.

To obtained the volume V_Ω, let us introduce a small pyramid, of which the base is ΔS shown in Fig.12 shadowed by hatching, and the vertex is center of gravity O, hence the height of the small pyramid is $g = r/\sqrt{2}$. Therefore when ΔT approaches zero, the volume of the small pyramid can be written as

$$\mathrm{dV}_\Omega = \frac{1}{3}\,g\,\mathrm{dS} = \frac{r}{3\sqrt{2}}\,\mathrm{dS} \quad . \tag{106}$$

Firstly substitute (102) in (106), secondly substitute (104) in the preceding result, then we get

$$\mathrm{V}_\Omega = 8\int_0^{\frac{\pi}{2}} \frac{r}{3\sqrt{2}} \left| \frac{\mathbf{U_{Ad}}+\mathbf{U_{Bd}}}{2} \times \mathbf{Q_A} \right| \mathrm{dt} = 8\frac{r}{3\sqrt{2}}\int_0^{\frac{\pi}{2}} \left| \frac{\mathbf{U_{Ad}}+\mathbf{U_{Bd}}}{2} \times \mathbf{Q_A} \right| \mathrm{dt} = \frac{r}{3\sqrt{2}}\,\mathrm{S}_\Omega$$

$$= 8\,\frac{r^3}{3\sqrt{2}}\,\left\{ \frac{\sqrt{2}-1}{2} + \mathrm{E}[\varphi_s, k_s] + \sqrt{2}\left(\Pi[-a, \varphi_s, k_s] + \Pi[b, \varphi_s, k_s] - \mathrm{F}[\varphi_s, k_s] \right) \right\} \; . \tag{107}$$

Hence the numerical result of (107) becomes

$$\mathrm{V}_\Omega = 3.281819487\cdots r^3 \quad . \tag{108}$$

The volume of a sphere of radius r is $4\pi r^3/3 = 4.188790205\cdots r^3$. Accordingly, V_Ω is smaller by about 21.65 % compared with the volume of a sphere of the same radius as the TCR.

Acknowledgements

I express my sincere gratitude to Dr. Christian Ucke for his faithful advice with relevant papers [4], [6], [7], and [8] which were written by many pioneers who have broken new ground in this field.

Epilogue

I have been interested in this theme ever since I saw the strange convex hull body among the public display items in the mathematics corner of Deutsch Museum (Bonn), which was shown on TV in an NHK (Japan Broadcasting Corporation) broadcast on Dec. 8. 2008. Through reading several papers, and from [4], it seems to me that the development of the TCR was not yet obtained as of 1995. I started to study the development with analytical geometry, and before not so long I came up against a brick wall. Fortunately around that time I found the suggestied idea on page at 23 in [1], and began to apply differential geometry to my studies. A few months later, on an internet web site, I found [5] with which I could look out over the general view of this theme. However, since I was using elliptic integrals in my work, I intended to approach this theme in a numerical way.
The numerical results and figures are worked out with MATEMATICA 4.1.

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